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AMPLITUDE AND PHASE DEMODULATION OF
FILTERED AM/PM SIGNALS

Edward Bedrosian

September 1986

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Prepared for

The Defense Advanced Research Projects Agency

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PREFACE

This Note reports on a portion of a one-year investigation of strategic communications performed for the Defense Advanced Research Projects Agency (DARPA) under RAND's National Defense Research Institute, a Federally Funded Research and Development Center sponsored by the Office of the Secretary of Defense. It presents an analysis of the distortion experienced by a hybrid AM/PM signal that has been filtered in the amplification stages of a typical receiver and then demodulated. The findings should be of interest to designers of communication and data link receivers and of communication satellite transponders. The research was performed in RAND's Engineering and Applied Sciences Department as an element of the Applied Science and Technology Program.

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I. INTRODUCTION AND SUMMARY

The AMBER (AM Broadcast Emergency Relay) system will provide voice and data communication capability throughout the continental United States for use during crises and for postattack reconstitution. Existing AM broadcast stations will be phase modulated to form a highly connected network with many alternative routes. It is clear that AMBER's successful implementation, and peacetime or crisis operation, will depend critically on its ability to operate without interfering with conventional broadcasts. That is, except in certain unusual circumstances, listeners using conventional AM receivers should not be aware of the auxiliary AMBER phase modulation.

An ideal AM receiver will pick up only the AM component of a hybrid AM/PM signal such as that transmitted by an AMBER station. Unfortunately, AM receivers typical of those in common use, even if they are well designed and properly operated, will also pick up the PM component. This comes about because the selectivity characteristics of typical AM receivers (as determined largely by their intermediate frequency amplifiers) allow phase variations of the input signal to be transformed into amplitude fluctuations that then appear in the receiver output. This is true even if the IF bandpass characteristics are symmetrical and the receiver is properly tuned (though these conditions minimize the interference). Only in the case of an ideal receiver with a uniform amplitude response and a linear phase characteristic is there no interference.

As a result, some interference in typical AM receivers is inevitable and a major goal for AMBER is to choose a data rate and phase modulation characteristics such that the resulting interference not be perceptible, even to a critical listener. Experimental and theoretical work to date suggests that signaling rates of about 150 symbols per second can be accommodated if the phase modulation waveforms are properly chosen.[1] Advanced modulation schemes capable of bandwidth compression, i.e., sending more bits per symbol, have also been investigated [2] to improve the capabilities of AMBER. The general

approach has been to take advantage of the more nearly ideal bandpass characteristics of typical AM receivers near the center of their bandpasses. Because of the poor audio response of typical AM receivers (and typical listeners) at low frequencies, this approach allows data rates of up to several hundred bits per second without interference.

To furnish a theoretical basis for future analyses and calculations, the output of an AM receiver to which the input is a hybrid AM/PM signal of the type to be encountered in AMBER is determined and presented here. The system to be analyzed is diagrammed in Fig. 1. An input signal is assumed to be simultaneously amplitude and phase modulated. The spectral response of the receiver is represented by a bandpass filter that may be unsymmetrical and mistuned. The demodulator output is the magnitude of the filtered input for an AM receiver, or its phase for a PM receiver. (Although the principal interest for AMBER is in AM receivers, the results for the output of a PM receiver are also presented for completeness.)

The results are in the form of an expansion in which the leading term is the linear (i.e., undistorted) reproduction of the input AM and PM and the successive terms represent the distortion. In both cases, the significant effects are given by the first few terms of the expansions because the applications of interest are those in which the

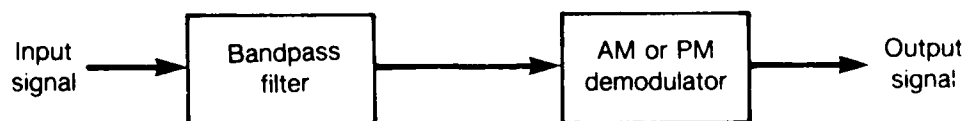


Fig. 1—Diagram of typical AM or PM receiver

interference is small.¹ Finally, a spectral analysis is performed on the output of an AM receiver. The leading DC, linear, and nonlinear terms of the output power spectrum are derived by assuming that the amplitude and phase modulating signals are uncorrelated zero-mean random gaussian processes. A similar analysis for a phase demodulator remains to be done.

This work is an extension of an earlier analysis [3] that considered the phase demodulation of a filtered PM signal. The results presented here are generalized to the use of a hybrid AM/PM signal and to the consideration of both phase and amplitude demodulation; the current approach differs in that it uses a generalized Taylor's series rather than the specialized expansion used before. Also, as an analytical convenience, it uses a relatively simple univariate expansion applied to an exponential form of AM. The results are then reduced to those that would have been obtained by making a more complicated bivariate expansion of the familiar linear form of AM. Finally, they are specialized to the case when the receiver is properly tuned and has a symmetric frequency response.

A recent extension [4] of Ref. 3 is related to the work presented here. It is a straightforward extension of the analytical approach used in Ref. 3 to the case of a hybrid AM/PM signal input to an off-tune receiver. The expansion of the output time function is presented for the case of a logarithmic (rather than linear) AM envelope demodulator and a linear phase detector. No spectral results are presented.

¹For well behaved input signals, the results are exact if the complete expansions are used. The fact that each expansion is a Taylor's series in which the entire linear (or undistorted) component of the output is given by the leading term suggests that the distortion, which is given by the remaining terms, is well approximated by the leading nonlinear term when the distortion is small.

II. THE MODULATED INPUT SIGNAL

When written as a real function of time, a hybrid AM/PM signal with a linear AM component becomes

$$s(t) = [1 + a(t)]\cos[2\pi f_0 t + \phi(t)], \quad |a(t)| \leq 1 \quad (1)$$

where f_0 is the carrier frequency, $a(t)$ is the amplitude modulating function, $\phi(t)$ is the phase modulating function, and the restriction on the magnitude of $a(t)$ prevents overmodulation.

If $a(t)$ and $\phi(t)$ are slowly varying in comparison with f_0 , the signal is said to be narrowband and its spectrum $S(f)$, which is given by the Fourier transform

$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-i2\pi ft} dt \quad (2)$$

contains most of its energy in the vicinity of $\pm f_0$. It is convenient for analysis to consider the complex signal

$$z_{lin}(t) = [1 + a(t)]e^{i\phi(t)}e^{i2\pi f_0 t}, \quad |a(t)| \leq 1 \quad (3)$$

which is approximately analytic and has negligible spectral content at negative frequencies.[5] The subscript *lin* denotes that the signal corresponds to conventional linear AM.

Generally speaking, if the spectrum of $a(t)$ is band limited to F_a , and $\phi(t)$ is constant, the spectrum of $z_{lin}(t)$ is limited to the band $\pm F_a$ about f_0 . However, when the signal is phase modulated, its spectrum extends indefinitely on either side of f_0 regardless of the spectral

extent of ϕ because of the transcendental nonlinear exponential operation. Fortunately, if the peak phase excursion of $\phi(t)$ is limited (to, say, $\pm\pi/2$ or less) and the spectrum of $\phi(t)$ is band limited to F_ϕ , the spectrum of the purely phase modulated signal (i.e., when $a(t)$ is constant) is largely confined to the band $\pm F_\phi$ about f_0 . If, in addition, the waveform of $\phi(t)$ is carefully chosen, the spectrum of the modulated signal will fall off rapidly at larger separations from f_0 . When modulated simultaneously by $a(t)$ and $\phi(t)$, the spectrum is extended somewhat by a convolution process to a value larger than that produced by either phase or amplitude modulation alone. Examples of this are shown in Fig. 21 of Ref. 1.

A hybrid AM/PM signal using exponential AM is given as a complex function by

$$z_{exp}(t) = e^{[a(t) + i\phi(t)]} e^{i2\pi f_0 t} = e^{h(t)} e^{i2\pi f_0 t} \quad (4)$$

where

$$h(t) = a(t) + i\phi(t) \quad (5)$$

and where the subscript exp denotes that the signal corresponds to exponential AM. Thus, whereas the signal given by (1) is a function of the two real variables $a(t)$ and $\phi(t)$, the signal given by (4) is a function of the single complex variable $h(t)$. Inasmuch as the expansion of the demodulator output is obtained using a Taylor's series (see Sec. IV), the use of the exponential form for the input results in a univariate expansion in terms of a single complex variable rather than a bivariate expansion in terms of two real variables. There is little difference in the complexity or difficulty of the two approaches for the first term or two in the expansion. However, higher-order terms in the bivariate expansion grow much more rapidly in complexity than those from the univariate expansion. In fact, general recursive relationships can

be developed in the univariate case to permit writing terms of any desired order explicitly; this does not seem to be possible with the bivariate expansion.

Of course the analytical simplicity of using the representation for exponential AM is of little value if the results so obtained cannot easily be related to those for linear AM, which is the form of modulation of practical interest. The relationship between linear and exponential AM can be seen by expanding the factor $\exp a(t)$ in (4) in a Taylor's series and then noting, from (3) that

$$z_{exp}(t) = [1 + a(t) + O(a^2)]e^{i\varphi(t)}e^{i2\pi f_s t} = z_{lin}(t) + O(a^2)e^{i\varphi(t)}e^{i2\pi f_s t} \quad (6)$$

where $O(a^2)$ denotes terms of order a^2 and higher. Thus, if $z_{exp}(t)$ is used as the input signal in Fig. 1, the output will contain the desired result represented by $z_{lin}(t)$ plus extraneous terms, all of which will involve factors $O(a^2)$. These factors are readily identified and deleted to give the desired result, as shown in Secs. V, VI, and VII.

III. THE FILTERED AND DEMODULATED OUTPUT SIGNAL

Consider an arbitrary linear filter having an impulse response $g(t)$ and a steady-state transfer function $G(f)$. These are then a Fourier pair

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{i2\pi ft} df \quad (7)$$

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-i2\pi ft} dt \quad (8)$$

where $g(t)$ is real and, for a physically realizable filter, vanishes when $t < 0$ (i.e., it is causal). $G(f)$ is complex, in general, with an even real part and an odd imaginary part; therefore

$$\overline{G(f)} = G(-f) \quad (9)$$

where the overbar denotes the complex conjugate.

The filter of interest here is a narrow bandpass filter. Such a filter is called bandpass because it passes a band of frequencies preferentially; i.e., the magnitude of its response $|G(f)|$ is large in the passband and small elsewhere. It is said to have a center frequency f_c , which may be the frequency at which the response is the greatest, or a frequency that characterizes the center of the passband in some sense. The choice is obvious for a symmetric filter but may be arbitrary for others; in general, it suffices that a center frequency be specified, however chosen. If the width of the passband is small (by any reasonable measure) in comparison with the center frequency, the filter is said to be narrowband.

Just as the bandwidth of the modulated signal is related to the frequency content of its modulating signal relative to carrier frequency, so also is the bandwidth of the bandpass filter related to the apparent frequency content of its impulse response relative to the center frequency. That is, the impulse response can be written

$$g(t) = r(t)\cos[2\pi f_c t + \theta(t)] \quad (10)$$

where $r(t)$ and $\theta(t)$ are real functions that vary slowly in comparison with f_c . Furthermore, just as the spectrum of the real modulated narrowband signal contains most of its energy in the vicinity of $\pm f_c$, so also does the transfer function of the narrow bandpass filter (having a real impulse response) contain most of its response in the vicinity of $\pm f_c$. Finally, just as the complex narrowband modulated signal $z(t)$ given by (3) is approximately analytic and has negligible spectral content at negative frequencies, so also would the complex impulse response

$$\gamma(t) = r(t)e^{i\theta(t)}e^{i2\pi f_c t} \quad (11)$$

of a narrow bandpass filter be approximately analytic and its transfer function $\Gamma(f)$ have negligible response at negative frequencies. (This suggests the possibility of an analytic filter, which is described in the Appendix.) It should be noted that the odd-conjugacy relationship for $G(f)$ given by (9) is not valid for $\Gamma(f)$. Rather,

$$\Gamma(f) \sim 0, \quad f < 0 \quad (12)$$

with the bulk of the response of $\Gamma(f)$ in the vicinity of f_c .

When the complex modulated signal given by (4) for a hybrid AM/PM signal having exponential AM is applied to the filter having the complex impulse response $\gamma(t)$ given by (11), the output becomes

$$\begin{aligned} x(t) &= \int_0^t \gamma(\tau) z(t - \tau) d\tau = \int_0^t r(\tau) e^{i\theta(\tau)} e^{i2\pi f_c \tau} e^{[a(t - \tau) + i\varphi(t - \tau)]} e^{i2\pi f_o(t - \tau)} d\tau \\ &= e^{i2\pi f_o t} \int_0^t \gamma'(\tau) e^{h(t - \tau)} d\tau \end{aligned} \quad (13)$$

where

$$\gamma'(t) = r(t) e^{i\theta(t)} e^{i2\pi(f_c - f_o)t} \quad (14)$$

and (5) has been used to introduce $h(t)$. The term $\gamma'(t)$ is the complex impulse response of a filter that is the lowpass equivalent of the mistuned narrow bandpass filter. It is seen from (11) and (14) that

$$\gamma'(t) = \gamma(t) e^{-i2\pi f_o t} \quad (15)$$

from which it follows that the transfer function $\Gamma'(f)$ of the lowpass equivalent to the mistuned narrow bandpass filter is given by

$$\Gamma'(f) = \int_0^t \gamma'(t) e^{-i2\pi f t} dt = \int_0^t \gamma(t) e^{-i2\pi(f + f_o)t} dt = \Gamma(f + f_o) \quad (16)$$

As a result,

$$\int_0^\infty \gamma'(t) dt = \Gamma(f_o) \quad (17)$$

The equivalent lowpass filter is not physically realizable, in general, but is approximately physically realizable for a nearly symmetric¹ narrow bandpass filter that is properly tuned.

When the filter output $x(t)$ given by (13) is amplitude or phase demodulated, the results are, respectively,

$$f(t) = \text{mag } x(t) = \sqrt{x(t) \bar{x}(t)} \quad (18)$$

or

$$f(t) = \text{arg } x(t) = \text{Im } \log x(t) \quad (19)$$

where Im denotes the imaginary part and the term $e^{i2\pi f_o t}$ in (13) has been suppressed. For convenience, the primes in (16) and (17) will be dropped henceforth and the quantities $\gamma(t)$ and $\Gamma(f)$ will be understood to refer to the impulse response and transfer function, respectively, of the normalized lowpass equivalent to the real narrow mistuned bandpass filter.

¹True symmetry of the transfer function of a physically realizable bandpass filter about its center frequency is not possible. However, symmetry can be well approximated if the bandwidth is sufficiently narrow. [6]

IV. TAYLOR'S SERIES EXPANSION

The demodulator outputs given by (18) and (19) are vector valued, i.e., each input modulation function $h(t, \omega)$, $\omega \in \Omega$ defines a value of the output that is a point or vector in Ω . Such vector valued functions are treated by Hille and Phillips [7] who show that under certain conditions the Taylor's series expansion of a vector valued function $f(y)$ is given by

$$f(y + h) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n f(y; h) \quad (20)$$

where $\delta^n f(y; h)$, the n th variation of $f(y)$ with increment h , is given by

$$\delta^n f(y; h) = \left. \frac{d^n}{d\zeta^n} f(y + \zeta h) \right|_{\zeta=0} \quad (21)$$

It is also shown that $\delta^n f(y; h)$ is a homogeneous polynomial of degree n in h and that, in particular, $\delta^n f(y; h)$ is linear in h .

Assuming differentiability, the demodulator outputs given by (18) and (19) will be expanded by applying the above. The expansions will be about the point $y = 0$ and h will be taken as the complex modulation function given by (5). That is, $f(y + h(t))$ in (20) will be identified with $f(t)$ in (18) and (19) by setting $y = 0$ and letting $h(t)$ be the variable of expansion rather than t , as is customarily done in the Taylor's series expansion of a time function. To avoid confusion with exponents, (20) and (21) will be written using subscripts rather than superscripts as

$$f(h) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta_n \quad (22)$$

and

$$\delta_n = \left. \frac{d^n}{d\xi^n} f(\xi h) \right|_{\xi=0} \quad (23)$$

where y has been set equal to zero.

V. AMPLITUDE DEMODULATION

The derivatives of the amplitude demodulator output (18) are best found recursively by starting with the square of (18). Writing ζh for t , yields

$$f^2(\zeta h) = x(\zeta h)\bar{x}(\zeta h) \quad (24)$$

The n th derivative of the left-hand side of (24) evaluated at $\zeta = 0$ becomes

$$\left. \frac{d^n}{d\zeta^n} f^2(\zeta h) \right|_{\zeta=0} = \sum_{r=0}^n {}_n C_r \frac{d^{n-r}}{d\zeta^{n-r}} f(\zeta h) \left. \frac{d^r}{d\zeta^r} f(\zeta h) \right|_{\zeta=0} = \sum_{r=0}^n {}_n C_r \delta_{n-r} \delta_r \quad (25)$$

where (23) was used to introduce δ_n and ${}_n C_r$ is the binomial coefficient.

$${}_n C_r = \frac{n!}{(n-r)!r!} \quad (26)$$

To facilitate differentiating $x(\zeta h)$, let (13) be written

$$x(\zeta h) \equiv \int \gamma e^{i\zeta h} \quad (27)$$

where (4) has been used for $z(t)$ and where a convolution integral is understood. The n th derivative of the right-hand side of (24) then becomes

$$\begin{aligned} \left. \frac{d^n}{d\zeta^n} x(\zeta h)\bar{x}(\zeta h) \right|_{\zeta=0} &= \left. \frac{d^n}{d\zeta^n} \int \gamma e^{i\zeta h} \int \gamma e^{i\zeta h} \right|_{\zeta=0} = \int \int \gamma_1 \gamma_2 \left. \frac{d^n}{d\zeta^n} e^{i\zeta(h_1 + h_2)} \right|_{\zeta=0} \\ &= \int \int \gamma_1 \gamma_2 (h_1 + h_2)^n \end{aligned} \quad (28)$$

where the subscripts identify the variables of integration. Equating (25) and (28) then yields

$$\sum_{r=0}^n C_r \delta_n \delta_r = \int \int \gamma_1 \bar{\gamma}_2 (h_1 + \bar{h}_2)^n \quad (29)$$

which is the basic recursive generating formula for the δ_n . Evaluating (29) for successive values of n leads to

$$\begin{aligned} \delta_0 &= |\Gamma_0| \\ \delta_1/\delta_0 &= I_1 \\ \delta_2/\delta_0 &= I_2 - (\delta_1/\delta_0)^2 = I_2 - I_1^2 \\ \delta_3/\delta_0 &= I_3 - 3(\delta_1/\delta_0)(\delta_2/\delta_0) = I_3 - 3I_1 I_2 + 3I_1^3 \\ \delta_4/\delta_0 &= I_4 - 4(\delta_1/\delta_0)(\delta_3/\delta_0) - 3(\delta_2/\delta_0)^2 = I_4 - 4I_1 I_3 + 18I_1^2 I_2 - 3I_2^2 - 15I_1^4 \\ \delta_5/\delta_0 &= I_5 - 5(\delta_1/\delta_0)(\delta_4/\delta_0) - 10(\delta_2/\delta_0)(\delta_3/\delta_0) \\ &= I_5 - 5I_1 I_4 + 30I_1^2 I_3 - 150I_1^3 I_2 + 45I_1 I_2^2 + 105I_1^5 - 10I_2 I_3 \end{aligned} \quad (30)$$

and so on, where (17) is used to evaluate δ_0 and where

$$I_n = \frac{1}{2|\Gamma_0|^2} \int \int \gamma_1 \gamma_2 (h_1 + h_2)^n, \quad n \geq 1 \quad (31)$$

The expansion for the amplitude demodulation of a filtered hybrid AM/PM signal having exponential AM (4) is then given by (22) using (30) for the δ_n and (31) for the I_n .

REDUCTION TO LINEAR AM

As mentioned in connection with (6), the results obtained using exponential AM can be reduced to those for linear AM by deleting terms $O(a^2)$ in the expansion. This can be done by noting that

$$\begin{aligned}
 (h_1 + \bar{h}_2)^n &= [(a_1 + i\varphi_1) + (a_2 - i\varphi_2)]^n \\
 &= [(a_1 + a_2) + i(\varphi_1 - \varphi_2)]^n = [i(\varphi_1 - \varphi_2)]^n + n(a_1 + a_2)[i(\varphi_1 - \varphi_2)]^{n-1} \\
 &\quad + \frac{n(n-1)}{2!}(a_1 + a_2)^2[i(\varphi_1 - \varphi_2)]^{n-2} \\
 &\quad + \frac{n(n-1)(n-2)}{3!}(a_1 + a_2)^3[i(\varphi_1 - \varphi_2)]^{n-3} + \dots
 \end{aligned} \tag{32}$$

It is apparent that terms $0(a^2)$ appear in all but the first two terms. When these undesired terms are deleted, there remain only the two leading terms and a term $n(n-1)a_1a_2 [i(\phi_1 - \phi_2)]^{n-2}$ from the third term¹; the fourth and succeeding terms vanish in their entirety. The result is

$$\begin{aligned}
 (h_1 + \bar{h}_2)^n &= [i(\varphi_1 - \varphi_2)]^n + n(a_1 + a_2)[i(\varphi_1 - \varphi_2)]^{n-1} + n(n-1)a_1a_2 [i(\varphi_1 - \varphi_2)]^{n-2} \\
 &= [n(a_1 + a_2) + i(\varphi_1 - \varphi_2)][i(\varphi_1 - \varphi_2)]^{n-1} + n(n-1)a_1a_2[i(\varphi_1 - \varphi_2)]^{n-2}
 \end{aligned} \tag{33}$$

Then, (31) becomes

$$I_n = \begin{cases} \frac{1}{2|\Gamma_0|^2} \int \int \gamma_1 \bar{\gamma}_2 (h_1 + \bar{h}_2) = \frac{1}{2|\Gamma_0|^2} \int \int \gamma_1 \bar{\gamma}_2 [(a_1 + a_2) + i(\varphi_1 - \varphi_2)], & n = 1 \\ \frac{1}{2|\Gamma_0|^2} \int \int \gamma_1 \bar{\gamma}_2 \{ [n(a_1 + a_2) + i(\varphi_1 - \varphi_2)][i(\varphi_1 - \varphi_2)]^{n-1} \\ \quad + n(n-1)a_1a_2 [i(\varphi_1 - \varphi_2)]^{n-2} \}, & n \geq 2 \end{cases} \tag{34}$$

¹The term containing the factor $a_1 a_2$, which might ordinarily be regarded as $0(a^2)$, is not $0(a^2)$ in this context because it results from the multiplication of terms containing the factors a_1 and a_2 , which are to be retained. The terms $0(a^2)$ to be deleted must actually contain factors a^2 , a^3 , etc.; these are the terms deleted in (32).

and the expansion for the amplitude demodulation of a filtered hybrid AM/PM signal having linear AM (3) is given by (22) using (30) for the δ_n and (34) for I_n .

SPECIALIZATION OF LINEAR AM TO A PROPERLY TUNED, NARROW, SYMMETRIC, BANDPASS FILTER

In this case, the impulse response $\gamma(t)$ of the equivalent lowpass filter is very nearly real so it is written as $r(t)$ to accord with (11). Then, noting (17) and substituting (34) into (30), the first few terms in the expansion become

$$\begin{aligned}\delta_0 &= \int r = \Gamma_0 \\ \delta_1 &= \int ra \\ \delta_2 &= \frac{1}{\Gamma_0} \left(\int r\varphi \right)^2 - \int r\varphi^2 \\ \delta_3 &= 3 \left[- \int ra\varphi^2 + \frac{2}{\Gamma_0} \int r\varphi \int ra\varphi - \frac{1}{\Gamma_0^2} \int ra \left(\int r\varphi \right)^2 \right] \quad (35)\end{aligned}$$

As expected, the DC term consists of the carrier (transposed to zero frequency by (16)) passed through the zero-frequency response of the equivalent lowpass filter. The linear term consists simply of the amplitude modulation filtered by the equivalent lowpass filter. The second-order term represents distortion produced by the phase modulation alone. The third- (and higher-) order terms include mixes between the amplitude and phase modulation. If there were no phase modulation (i.e., $\phi(t) = 0$), then only the linear term would remain.

VI. PHASE DEMODULATION

The phase demodulator output (19) is found most conveniently by expanding $\log x(t)$ and then taking the imaginary part. Thus, instead of (22), the expansion is given by

$$f(h) = \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{Im} \delta_n \quad (36)$$

where δ_n is again given by (23).

The first two terms of (36) can be obtained directly. Thus

$$\delta_0 = f(\zeta h) \Big|_{\zeta=0} = \log x(\zeta h) \Big|_{\zeta=0} = \log \int \gamma e^{\zeta h} \Big|_{\zeta=0} = \log \int \gamma = \log \Gamma_0 \quad (37)$$

and

$$\delta_1 = \frac{d}{d\zeta} f(\zeta h) \Big|_{\zeta=0} = \frac{d}{d\zeta} \log x(\zeta h) \Big|_{\zeta=0} = \frac{\int \gamma h e^{\zeta h}}{\int \gamma e^{\zeta h}} \Big|_{\zeta=0} = \frac{\int \gamma h}{\int \gamma} = \frac{1}{\Gamma_0} \int \gamma h \quad (38)$$

where (17) has been used. For the higher derivatives, it is better to start with the first derivative

$$\frac{d}{d\zeta} f(\zeta h) = \frac{d}{d\zeta} \log x(\zeta h) = \frac{\frac{d}{d\zeta} x(\zeta h)}{x(\zeta h)}$$

and write it as

$$x(\zeta h) \frac{d}{d\zeta} f(\zeta h) = \frac{d}{d\zeta} x(\zeta h) \quad (39)$$

Differentiating $n - 1$ times more then yields

$$\sum_{r=0}^{n-1} {}_{n-1}C_r \frac{d^r}{d\zeta^r} x(\zeta h) \frac{d^{n-r}}{d\zeta^{n-r}} f(\zeta h) = \frac{d^n}{d\zeta^n} x(\zeta h) \quad (40)$$

Now

$$\frac{d^n}{d\zeta^n} x(\zeta h) = \frac{d^n}{d\zeta^n} \int \gamma e^{i\zeta h} = \int \gamma h^n e^{i\zeta h} \quad (41)$$

so using (23) to introduce δ_n and setting $\zeta = 0$ permits (40) to be written

$$\sum_{r=0}^{n-1} {}_{n-1}C_r \int \gamma h^r \delta_{n-r} = \int \gamma h^n \quad (42)$$

or

$$\delta_n \int \gamma + \sum_{r=1}^{n-1} {}_{n-1}C_r \int \gamma h^r \delta_{n-r} = \int \gamma h^n \quad (43)$$

where the first term has been extracted from the sum. Noting (17) and transposing then yields

$$\delta_n = \frac{1}{\Gamma_0} \left[\int \gamma h^n - \sum_{r=1}^{n-1} {}_{n-1}C_r \delta_{n-r} \int \gamma h^r \right], \quad n \geq 2 \quad (44)$$

Letting

$$J_n \equiv \frac{1}{\Gamma_0} \int \gamma h^n, \quad n \geq 1 \quad (45)$$

the leading terms in the expansion become, from (37), (38), and (44),

$$\delta_0 = \log \Gamma_0$$

$$\delta_1 = J_1$$

$$\delta_2 = J_2 - \delta_1 J_1 = J_2 - J_1^2$$

$$\delta_3 = J_3 - 2\delta_2 J_1 - \delta_1 J_2 = J_3 - 3J_1 J_2 + 2J_1^3$$

$$\delta_4 = J_4 - 3\delta_3 J_1 - 3\delta_2 J_2 - \delta_1 J_3 = J_4 - 4J_1 J_3 + 12J_1^2 J_2 - 3J_2^2 - 6J_1^4$$

$$\delta_5 = J_5 - 4\delta_4 J_1 - 6\delta_3 J_2 - 4\delta_2 J_3 - \delta_1 J_4$$

$$= J_5 - 5J_1 J_4 + 20J_1^2 J_3 - 60J_1^3 J_2 + 30J_1 J_2^2 - 10J_2 J_3 + 24J_1^5 \quad (46)$$

and so on. The expansion for the phase demodulation of a filtered hybrid AM/PM signal having exponential AM (4) is then given by (36) using (46) for the δ_n and (45) for the J_n .

REDUCTION TO LINEAR AM

As was done with amplitude demodulation, the reduction to linear AM is accomplished by deleting terms $O(a^2)$ in the expansion. Now

$$h^n = (a + i\varphi)^n = (i\varphi)^n + na(i\varphi)^{n-1} + \frac{n(n-1)}{2!} a^2(i\varphi)^{n-2} + \dots \quad (47)$$

Deleting terms $O(a^2)$ and substituting (47) in (45) then yields

$$J_n = \frac{1}{\Gamma_0} \int \gamma(na + i\varphi)(i\varphi)^{n-1}, \quad n \geq 1 \quad (48)$$

Thus, the expansion for the phase demodulation of a filtered hybrid AM/PM signal having linear AM (3) is given by (36) using (46) for the δ_n and (48) for the J_n . If the terms in $a(t)$ are deleted to reduce the results to the case of pure PM, the expansion agrees with the earlier one given in Ref. 3. The first few terms are

$$\begin{aligned} \text{Im } \delta_0 &= \text{Im } \log \Gamma_0 \\ \text{Im } \delta_1 &= \text{Re } \frac{1}{\Gamma_0} \int \gamma\varphi \\ \text{Im } \delta_2 &= - \text{Im } \left[\frac{1}{\Gamma_0} \int \gamma\varphi^2 - \frac{1}{\Gamma_0^2} \left(\int \gamma\varphi \right)^2 \right] \\ \text{Im } \delta_3 &= - \text{Re } \left[\frac{1}{\Gamma_0} \int \gamma\varphi^3 - \frac{3}{\Gamma_0^2} \int \gamma\varphi \int \gamma\varphi^2 + \frac{2}{\Gamma_0^3} \left(\int \gamma\varphi \right)^3 \right] \end{aligned} \quad (49)$$

where Re denotes the real part.

SPECIALIZATION OF LINEAR AM TO A PROPERLY TUNED, NARROW, SYMMETRIC, BANDPASS FILTER

In this case, the impulse response $r(t)$ of the equivalent lowpass filter is very nearly real so it is written as $r(t)$ to accord with (11). Then, noting (17) and substituting (48) into (46), the first few terms in the expansion become

$$Im \delta_0 = 0$$

$$Im \delta_1 = \frac{1}{\Gamma_0} \int r \varphi$$

$$Im \delta_2 = \frac{2}{\Gamma_0} \int r a \varphi - \frac{2}{\Gamma_0^2} \int r a \int r \varphi$$

$$Im \delta_3 = -\frac{1}{\Gamma_0} \int r \varphi^3 - \frac{3}{\Gamma_0^2} \left[2 \int r a \int r a \varphi - \int r \varphi \int r \varphi^2 \right] \\ + \frac{2}{\Gamma_0^3} \left[3 \left(\int r a \right)^2 \int r \varphi - \left(\int r \varphi \right)^3 \right] \quad (50)$$

As expected, the DC term vanishes because the zero-frequency response Γ_0 of the equivalent lowpass filter is now real. The linear term consists simply of the phase modulation filtered by the equivalent lowpass filter. The distortion terms include mixes between the amplitude and phase modulation. If there were no amplitude modulation (i.e., $a(t) = 0$), the second-order term $Im \delta_2$ (and, probably, all higher-order even terms) would disappear but the third order term $Im \delta_3$ (and, probably, all higher-order odd terms) would retain intrinsic PM distortion. Only in the case of a white filter (infinitely wide, uniform-amplitude passband--in effect, an all-pass filter but with a possible time delay) does the distortion disappear completely. This can be seen from $Im \delta_3$ in (50) by taking $r(t)$ as the Dirac delta function, which is the impulse response of such a filter. Then,

$$Im \delta_3 = -\varphi^3 + 3\varphi\varphi^2 - 2\varphi\varphi\varphi = 0 \quad (51)$$

as indicated above.

VII. SPECTRAL ANALYSIS OF AMPLITUDE DEMODULATOR OUTPUT

The output of an amplitude demodulator operating on a filtered hybrid AM/PM signal was derived in Sec. V. The leading terms in the power spectrum of that output are developed in this section by assuming that the amplitude and phase modulation components (5) of the basic hybrid AM/PM signal are uncorrelated, zero-mean, gaussian random processes. The results are first developed for the case of the complex exponential AM (4); they are then reduced to the case of linear AM (3), and finally specialized to the case where the filter is symmetric and properly tuned. The expansion for the amplitude demodulator output is given by (22) using (30) for the δ_n and (31) for the I_n . The equivalent lowpass filter parameters are given by (15) and (16), which are related according to

$$\int \gamma_t e^{i2\pi ft} dt = \Gamma_{zf}, \quad \int \bar{\gamma}_t e^{i2\pi ft} dt = \bar{\Gamma}_{zf} \quad (52)$$

where, as before, the overbar denotes the complex conjugate and subscripts are used to identify the variables.

AUTOCORRELATION FUNCTIONS AND POWER SPECTRA

The power spectrum $W(f)$ of a function $f(t)$ is found by forming the autocorrelation function

$$R_\tau = E f_t f_{t+\tau} \quad (53)$$

where E denotes the expectation operator, and then taking its Fourier transforms F

$$W_f = FR_\tau = \int_{-\infty}^{\infty} R_\tau e^{-i2\pi f\tau} d\tau \quad (54)$$

where R and W are a Fourier pair. Let

$$\delta_i \otimes \delta_j = FE \delta_i(t) \delta_j(t + \tau) \quad (55)$$

where the subscripts are again used in connection with the δ s to identify the orders of the terms. Then, from (22),

$$W(f) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i!j!} \delta_i \otimes \delta_j \quad (56)$$

which is the desired termwise expansion of the output power spectrum. It should be noted from (55) that if

$$\delta_i \otimes \delta_j = W_{ij}(f) \quad (57)$$

then

$$\delta_j \otimes \delta_i = FE \delta_j(t) \delta_i(t + \tau) = FE \delta_i(t) \delta_j(t - \tau) = FR_{ij}(-\tau) = W_{ij}(-f) \quad (58)$$

In forming the various autocorrelation functions, it will be necessary to consider the expected values of products of the h factors (5). When a and ϕ are uncorrelated, zero-mean, gaussian random processes, it follows that

$$\begin{aligned} Ea &= E\varphi = Ea\varphi = 0 \\ Eh_1 h_2 &= E(a_1 + i\varphi_1)(a_2 + i\varphi_2) = Ea_1 a_2 - E\varphi_1 \varphi_2 = R_{2-1}^a - R_{2-1}^\varphi = R_{2-1}^- \\ Eh_1 \bar{h}_2 &= E(a_1 + i\varphi_1)(a_2 - i\varphi_2) = Ea_1 a_2 + E\varphi_1 \varphi_2 = R_{2-1}^a + R_{2-1}^\varphi = R_{2-1}^+ \\ E\bar{h}_1 h_2 &= E(a_1 - i\varphi_1)(a_2 + i\varphi_2) = Ea_1 a_2 + E\varphi_1 \varphi_2 = R_{2-1}^a + R_{2-1}^\varphi = R_{2-1}^+ \\ E\bar{h}_1 \bar{h}_2 &= E(a_1 - i\varphi_1)(a_2 - i\varphi_2) = Ea_1 a_2 - E\varphi_1 \varphi_2 = R_{2-1}^a - R_{2-1}^\varphi = R_{2-1}^- \end{aligned} \quad (59)$$

where (53) has been used to introduce R . The superscripts in (59) identify the functions a and ϕ whose autocorrelation functions are being taken; the superscripts $+$ and $-$ refer to the two arithmetic combinations of these autocorrelation functions. The differences expressed as subscripts refer to the differences in the arguments of the variables in question.

To consider products such as $(h_1 + \bar{h}_2)^m (h_3 + \bar{h}_4)^n$, which arise when products $I_m I_n$ from (31) are formed, it is noted that for an even number of random variables

$$E f_1 f_2 \cdots f_{2N} = \sum_{\substack{\text{all} \\ \text{pairs}}} E f_i f_j E f_k f_l \cdots E f_m f_n \quad (60)$$

where $(i,j), (k,l), \dots, (m,n)$ are N pairs of integers selected from $1, 2, \dots, 2N$ and the summation extends over all the $1 \cdot 3 \cdot 5 \cdots (2N - 1)$ possible pairings. When the number of random variables is odd,

$$E f_1 f_2 \cdots f_{2N-1} = 0 \quad (61)$$

In applying (60) it is convenient to consider $h_1 + \bar{h}_2$ as the basic, zero-mean, gaussian random process and to note, from (59) that

$$E(h_1 + \bar{h}_2)(h_3 + \bar{h}_4) = R_{3-1} + R_{4-1} + R_{3-2} + R_{4-2} \quad (62)$$

For convenience, paralleling the notation in (59) let

$$W_f' = W_f^a \cdot W_f^{\phi} \quad FR_f' \quad (63)$$

Additional useful formulations obtained from (54) are

$$FR_{\tau,a} = W_f e^{i2\pi/a} \quad (64)$$

$$FR_{\tau,a} R_{\tau,c,d} = \int W_p W_f e^{i2\pi[\rho a + \rho b + (f-\rho)c + (f-\rho)d]} d\rho \quad (65)$$

CROSS TERMS

When the cross terms $\delta_i \otimes \delta_j$ are formed in (56), it is seen from (30), (31), and (55) that the resulting integrals will always have products of h s in their integrands of the form

$$(h_1 + \bar{h}_2)^a (h_3 + \bar{h}_4)^b (h_5 + \bar{h}_6)^c \cdots \quad (66)$$

where

$$a + b + c + \cdots = i + j \quad (67)$$

Hence, from (61) it is seen that all cross terms for which $i + j$ is odd will vanish when the expected value is taken. When $i + j$ is even, the following cross terms can be identified:

$$i + j = 0: \quad \delta_0 \otimes \delta_0 = FE |\Gamma_0|^2 = |\Gamma_0|^2 \delta(0) \quad \underline{\text{Leading DC term}} \quad (68)$$

where $\delta(0)$ is the Dirac delta function

$$i + j = 2: \quad \delta_0 \otimes \delta_2, \delta_2 \otimes \delta_0 \quad \text{Second-order DC terms} \quad (69)$$

$$\delta_1 \otimes \delta_1 \quad \text{Leading linear term} \quad (70)$$

$$i + j = 4: \delta_0 \otimes \delta_4, \delta_4 \otimes \delta_0 \quad \text{Third-order DC terms} \quad (71)$$

$$\delta_1 \otimes \delta_3, \delta_3 \otimes \delta_1 \quad \text{Second-order linear terms} \quad (72)$$

$$\delta_2 \otimes \delta_2 \quad \text{Leading nonlinear term (plus others)} \quad (73)$$

The DC terms (68), (69), (71), and portions of (73) stem from pairings in (66) according to (60) of which none display the form $f_t f_{t \pm \tau}$. The second-order linear terms (72) are so labeled because only one of the pairings is of the form $f_t f_{t \pm \tau}$; thus, they represent corrections to (70) which is the leading linear term consisting of a single $f_t f_{t \pm \tau}$ pairing. The analytic problem thus reduces to evaluating $\delta_1 \otimes \delta_1$ in (70) to obtain the leading linear term and the appropriate portions of $\delta_2 \otimes \delta_2$ in (73) to obtain the leading nonlinear terms.

EXPONENTIAL AM

Leading Linear Term

Using (30) for δ_1 and (31) for I_1 in (70) leads to

$$\begin{aligned} \delta_1 \otimes \delta_1 &= FE \delta_0^2 I_1(t) I_2(t + \tau) \\ &= FE \frac{\delta_0^2}{4 |\Gamma_0|^4} \int \int \gamma_u \gamma_v (h_{t-u} + h_{t-v}) du dv \int \int \gamma_u \gamma_v (h_{t+\tau-u} + h_{t+\tau-v}) du dv \quad (74) \end{aligned}$$

Expanding and applying (62) then yields

$$\delta_1 \otimes \delta_1 = \frac{1}{4|\Gamma_0|^2} F \int \int \int \int \gamma_u \bar{\gamma}_v \gamma_w \bar{\gamma}_x (R_{\tau-w+u}^- + R_{\tau-x+u}^+ + R_{\tau-w+v}^+ + R_{\tau-x+v}^-) du dv dw dx \quad (75)$$

The Fourier transform of the autocorrelation functions is found by using (64), which leads to

$$\begin{aligned} \delta_1 \otimes \delta_1 = \frac{1}{4|\Gamma_0|^2} \int \int \int \int \gamma_u \bar{\gamma}_v \gamma_w \bar{\gamma}_x [W_f^- e^{i2\pi f(-w+u)} + W_f^+ e^{i2\pi f(-x+u)} \\ + W_f^+ e^{i2\pi f(-w+v)} + W_f^- e^{i2\pi f(-x+v)}] du dv dw dx \end{aligned} \quad (76)$$

The transfer function of the equivalent lowpass filter is introduced by applying (52) to yield

$$\begin{aligned} \delta_1 \otimes \delta_1 &= \frac{1}{4|\Gamma_0|^2} [W_f^- \Gamma_{-f} \bar{\Gamma}_0 \Gamma_f \bar{\Gamma}_0 + W_f^+ \Gamma_{-f} \bar{\Gamma}_0 \Gamma_0 \bar{\Gamma}_{-f} + W_f^+ \Gamma_0 \bar{\Gamma}_f \Gamma_f \bar{\Gamma}_0 + W_f^- \Gamma_0 \bar{\Gamma}_f \Gamma_0 \bar{\Gamma}_{-f}] \\ &= \frac{1}{4|\Gamma_0|^2} [W_f^+ |\Gamma_0|^2 (|\Gamma_f|^2 + |\Gamma_{-f}|^2) + 2W_f^- \text{Re} \bar{\Gamma}_0^2 \Gamma_f \Gamma_{-f}] \end{aligned} \quad (77)$$

Finally, the result is expressed in terms of the power spectra of the amplitude and phase modulating signals themselves by using (63) to obtain

$$\delta_1 \otimes \delta_1 = \frac{I}{4|\Gamma_0|^2} \left\{ \frac{W_f^a}{W_f^p} \right\} |\bar{\Gamma}_0 \Gamma_f \pm \Gamma_0 \bar{\Gamma}_{-f}|^2 \quad (78)$$

where the matrix-multiplication-like notation

$$\left\{ \frac{W_f^0}{W_f^1} \right\} |\bar{\Gamma}_0 \Gamma_f \pm \Gamma_0 \bar{\Gamma}_{-f}|^2 = W_f^0 |\bar{\Gamma}_0 \Gamma_f + \Gamma_0 \bar{\Gamma}_{-f}|^2 + W_f^1 |\bar{\Gamma}_0 \Gamma_f - \Gamma_0 \bar{\Gamma}_{-f}|^2 \quad (79)$$

has been introduced for convenience. Equation (78) is the desired expression for the leading linear term in the output power spectrum (56).

Leading Nonlinear Term

The derivation of the leading nonlinear term $\delta_2 \otimes \delta_2$ parallels that of the leading linear term $\delta_1 \otimes \delta_1$ above except that many tedious algebraic manipulations are involved. For simplicity, the various steps will be outlined and intermediate results indicated without detailing the intervening steps.

Using (30) for δ_2 in (73) leads to

$$\begin{aligned} \delta_2 \otimes \delta_2 &= FE \delta_0^2 [I_2(t) - I_1^2(t)][I_2(t + \tau) - I_1^2(t + \tau)] \\ &= FE \delta_0^2 [I_2(t) I_2(t + \tau) - I_2(t) I_1^2(t + \tau) - I_1^2(t) I_2(t + \tau) + I_1^2(t) I_1^2(t + \tau)] \\ &= T_1 - (T_2 + T_3) + T_4 \end{aligned} \quad (80)$$

The four terms in (80) have been identified, respectively, as T_1 , T_2 , T_3 , and T_4 for convenience and will be evaluated in succession. See (82), (90), and (95) for the resulting formulations for T_1 , $T_2 + T_3$, and T_4 .

Nonlinear Term T_1 . Using (31) for I_2 in (80) leads to

$$T_1 = FE \frac{\delta_0^2}{4 |\Gamma_0|^4} \iint \gamma_u \bar{\gamma}_v (h_{t-u} + \bar{h}_{t-v})^2 du dv \iint \gamma_u \bar{\gamma}_v (h_{t+\tau-u} + \bar{h}_{t+\tau-v})^2 du dv \quad (81)$$

The expected value of the product of the four terms represented by the parentheses in the integrands leads to the sum of three products of pairs of expected values according to (60). It is seen that two of these are identical leading to

$$T_1 = \frac{1}{4|\Gamma_0|^2} F \iiint \gamma_u \bar{\gamma}_v \gamma_w \bar{\gamma}_x [E(h_{t-u} + \bar{h}_{t-v})^2 E(h_{t+\tau-w} + \bar{h}_{t+\tau-x})^2 + 2E^2(h_{t-u} + \bar{h}_{t-v})(h_{t+\tau-w} + \bar{h}_{t+\tau-x})] du dv dw dx \quad (82)$$

The first term in (82) is clearly a third-order DC term because neither expected value leads to a pairing of the form $f_t f_{t+\tau}$. Thus, only the second term need be considered. Applying (62) to (82) then yields

$$T_1 = \frac{1}{2|\Gamma_0|^2} \iiint \gamma_u \bar{\gamma}_v \gamma_w \bar{\gamma}_x F(R_{\tau-w+u}^- + R_{\tau-x+u}^+ + R_{\tau-w+v}^+ + R_{\tau-x+v}^-)^2 du dv dw dx \quad (83)$$

Expanding the parenthesis leads to sixteen products of autocorrelation functions whose Fourier transforms can be found by using (65). After extensive manipulation and the use of (52) to introduce the transfer function of the equivalent lowpass filter, the result becomes

$$T_1 = \frac{1}{2|\Gamma_0|^2} \int \left\{ W_\rho^- W_{f-\rho}^- 2Re(\bar{\Gamma}_0^2 \Gamma_f \Gamma_{-f} + \Gamma_\rho \Gamma_{-\rho} \bar{\Gamma}_{f-\rho} \bar{\Gamma}_{\rho-f}) + W_\rho^- W_{f-\rho}^+ 4Re\bar{\Gamma}_0(\Gamma_{-f} \bar{\Gamma}_{\rho-f} \Gamma_\rho + \Gamma_{-\rho} \bar{\Gamma}_{f-\rho} \Gamma_f) + W_\rho^+ W_{f-\rho}^+ [|\Gamma_0|^2(|\Gamma_f|^2 + |\Gamma_{-f}|^2) + 2|\Gamma_{-\rho}|^2|\Gamma_{f-\rho}|^2] \right\} d\rho \quad (84)$$

Finally, the result is expressed in terms of the power spectra of the amplitude and phase modulating signals themselves by using (63) to obtain, after further manipulation,

$$T_1 = \frac{1}{2|\Gamma_0|^2} \int \left\{ \frac{W_\rho^a}{W_\rho^e} \frac{W_{f-\rho}^a}{W_{f-\rho}^e} \right\} |\Gamma_0 \bar{\Gamma}_f + \Gamma_{-f} \bar{\Gamma}_0 + \Gamma_{\rho-f} \bar{\Gamma}_\rho + \Gamma_{-\rho} \bar{\Gamma}_{f-\rho}|^2 d\rho \quad (85)$$

where the notation is an extension of that used in (79).

Nonlinear Terms T_2 and T_3 . It can be seen from (57) and (58) that T_2 and T_3 in (80) are a mirror-frequency pair. Therefore, T_2 will be evaluated first and T_3 found by writing $-f$ for f in T_2 . Using (31) for I_1 and I_2 leads to

$$T_2 = FE \frac{\delta_0^2}{8|\Gamma_0|^6} \int \int \gamma_u \bar{\gamma}_v (h_{t-u} + \bar{h}_{t-v})^2 du dv \left[\int \int \gamma_u \bar{\gamma}_v (h_{t+\tau-u} + \bar{h}_{t+\tau-v}) du dv \right]^2 \quad (86)$$

Using (60) in the same fashion as with (81) leads to

$$T_2 = \frac{1}{8|\Gamma_0|^4} F \int \cdots \int \gamma_u \bar{\gamma}_v \gamma_w \bar{\gamma}_x \gamma_y \bar{\gamma}_z [E(h_{t-u} + \bar{h}_{t-v})^2 E(h_{t+\tau-w} + \bar{h}_{t+\tau-x})(h_{t+\tau-y} + \bar{h}_{t+\tau-z}) \\ + 2E(h_{t-u} + \bar{h}_{t-v})(h_{t+\tau-w} + \bar{h}_{t+\tau-x})E(h_{t-u} + \bar{h}_{t-v})(h_{t+\tau-y} + \bar{h}_{t+\tau-z})] du \cdots dz \quad (87)$$

As in (82), only the second term need be considered because the first term is a third-order DC term. Applying (62) to (87) then yields

$$T_2 = \frac{1}{4|\Gamma_0|^4} F \int \cdots \int \gamma_u \bar{\gamma}_v \gamma_w \bar{\gamma}_x \gamma_y \bar{\gamma}_z (R_{\tau-w+u}^- + R_{\tau-x+u}^+ + R_{\tau-w+v}^+ + R_{\tau-x+v}^-) \\ (R_{\tau-y+u}^- + R_{\tau-z+u}^+ + R_{\tau-y+v}^+ + R_{\tau-z+v}^-) du \cdots dz \quad (88)$$

Expanding the product of the parentheses leads to sixteen products of autocorrelation functions whose Fourier transforms can be found by using (65). After some manipulation, the use of (52) to introduce the

transfer function of the equivalent lowpass filter, and invoking the mirror-frequency property of T_2 and T_3 , the result becomes

$$\begin{aligned}
 T_2 + T_3 = & \frac{1}{2|\Gamma_0|^4} \int \left\{ W_\rho^- W_{f-\rho}^- \text{Re}(\bar{\Gamma}_0^3 \Gamma_\rho \Gamma_{f-\rho} \Gamma_{-f} + 2|\Gamma_0|^2 \Gamma_\rho \Gamma_{-\rho} \bar{\Gamma}_{f-\rho} \bar{\Gamma}_{\rho-f} + \Gamma_0^3 \bar{\Gamma}_{-\rho} \bar{\Gamma}_{\rho-f} \bar{\Gamma}_f) \right. \\
 & + 2W_\rho^- W_{f-\rho}^+ \text{Re}(|\Gamma_0|^2 \bar{\Gamma}_0 \Gamma_\rho \bar{\Gamma}_{\rho-f} \Gamma_{-f} + \bar{\Gamma}_0^2 |\Gamma_{f-\rho}|^2 \Gamma_\rho \Gamma_{-\rho} + \Gamma_0^2 |\Gamma_{\rho-f}|^2 \bar{\Gamma}_\rho \bar{\Gamma}_{-\rho} + |\Gamma_0|^2 \bar{\Gamma}_0 \bar{\Gamma}_{-\rho} \Gamma_{f-\rho} \bar{\Gamma}_f) \\
 & \left. + W_\rho^+ W_{f-\rho}^+ |\Gamma_0|^2 [2|\Gamma_\rho|^2 |\Gamma_{\rho-f}|^2 + \text{Re}(\Gamma_0 \bar{\Gamma}_{-\rho} \bar{\Gamma}_{\rho-f} \Gamma_{-f} + \bar{\Gamma}_0 \Gamma_\rho \Gamma_{f-\rho} \bar{\Gamma}_f)] \right\} d\rho \quad (89)
 \end{aligned}$$

Finally, the result is expressed in terms of the power spectra of the amplitude and phase modulating signals themselves by using (63) to obtain, after further manipulation,

$$\begin{aligned}
 T_2 + T_3 = & \frac{1}{2|\Gamma_0|^4} \int \left\{ \begin{array}{cc} W_\rho^a & W_{f-\rho}^a \\ 2W_\rho^a & W_{f-\rho}^a \\ W_\rho^\varphi & W_{f-\rho}^\varphi \end{array} \right\} \\
 & \text{Re}(\Gamma_{-f} \bar{\Gamma}_0 + \Gamma_0 \bar{\Gamma}_f + \Gamma_{-\rho} \Gamma_{f-\rho} + \Gamma_{\rho-f} \bar{\Gamma}_\rho)(\Gamma_0 \bar{\Gamma}_{-\rho} + \Gamma_\rho \bar{\Gamma}_0)(\Gamma_0 \bar{\Gamma}_{\rho-f} + \Gamma_{f-\rho} \bar{\Gamma}_0) d\rho \quad (90)
 \end{aligned}$$

where the notation is again an extension of that used in (79).

Nonlinear Term T_4 . Using (31) for I_1 in (80) leads to

$$T_4 = FE \frac{\delta_0^2}{16|\Gamma_0|^8} \left[\int \int \gamma_u \bar{\gamma}_v (h_{t-u} + \bar{h}_{t-v}) du dv \right]^2 \left[\int \int \gamma_u \bar{\gamma}_v (h_{t+\tau-u} + \bar{h}_{t+\tau-v}) du dv \right]^2 \quad (91)$$

Using (60) in the same fashion as with (81) leads to

$$\begin{aligned}
 T_4 = & \frac{1}{16|\Gamma_0|^6} F \int \cdots \int \gamma_u \bar{\gamma}_v \gamma_w \bar{\gamma}_x \gamma_y \bar{\gamma}_z \gamma_\alpha \bar{\gamma}_\beta \\
 & [E(h_{t-u} + \bar{h}_{t-v})(h_{t-w} + \bar{h}_{t-x})E(h_{t+\tau-y} + \bar{h}_{t+\tau-z})(h_{t+\tau-\alpha} + \bar{h}_{t+\tau-\beta}) \\
 & + E(h_{t-u} + \bar{h}_{t-v})(h_{t+\tau-y} + \bar{h}_{t+\tau-z})E(h_{t-w} + \bar{h}_{t-x})(h_{t+\tau-\alpha} + \bar{h}_{t+\tau-\beta}) \\
 & + E(h_{t-u} + \bar{h}_{t-v})(h_{t+\tau-\alpha} + \bar{h}_{t+\tau-\beta})E(h_{t-w} + \bar{h}_{t-x})(h_{t+\tau-y} + \bar{h}_{t+\tau-z})] du \cdots d\beta
 \end{aligned} \tag{92}$$

As in (82), the first term is a third-order DC term. Also, it can be seen by symmetry of the variables that the second and third terms are identical. Applying (62) to (92) then yields

$$\begin{aligned}
 T_4 = & \frac{1}{8|\Gamma_0|^6} F \int \cdots \int \gamma_u \bar{\gamma}_v \gamma_w \bar{\gamma}_x \gamma_y \bar{\gamma}_z \gamma_\alpha \bar{\gamma}_\beta (R_{\tau-\alpha+u} + R_{\tau-\beta+u} + R_{\tau-\alpha+v} + R_{\tau-\beta+v}) \\
 & (R_{\tau-y+w} + R_{\tau-z+w} + R_{\tau-y+x} + R_{\tau-z+x}) du \cdots d\beta
 \end{aligned} \tag{93}$$

Expanding the product of the parentheses leads to sixteen products of auto-correlation functions whose Fourier transforms can be found by using (65). After some manipulation, and the use of (52) to introduce the transfer function of the equivalent lowpass filter, the result becomes

$$\begin{aligned}
 T_4 = & \frac{1}{8|\Gamma_0|^6} \int [W_\rho^- W_{f-\rho}^- 2\text{Re} \Gamma_\rho \Gamma_{-\rho} (\bar{\Gamma}_0^4 \Gamma_{\rho-f} \Gamma_{f-\rho} + |\Gamma_0|^4 \bar{\Gamma}_{\rho-f} \bar{\Gamma}_{f-\rho}) \\
 & + W_\rho^- W_{f-\rho}^+ 4\text{Re} |\Gamma_0|^2 \bar{\Gamma}_0^2 \Gamma_\rho \Gamma_{-\rho} (|\Gamma_{\rho-f}|^2 + |\Gamma_{f-\rho}|^2) \\
 & + W_\rho^+ W_{f-\rho}^+ |\Gamma_0|^4 (|\Gamma_{\rho-f}|^2 + |\Gamma_{f-\rho}|^2) (|\Gamma_\rho|^2 + |\Gamma_{-\rho}|^2)] d\rho
 \end{aligned} \tag{94}$$

Finally, the result is expressed in terms of the power spectra of the

amplitude and phase modulating signals themselves by using (63) to obtain, after further manipulation,

$$T_4 = \frac{1}{8|\Gamma_0|^6} \int \left\{ \begin{array}{cc} W_\rho^a & W_{f-\rho}^a \\ 2W_\rho^a & W_{f-\rho}^a \\ W_\rho^a & W_{f-\rho}^a \end{array} \right\} |\bar{\Gamma}_0 \bar{\Gamma}_{-\rho} + \Gamma_0 \bar{\Gamma}_\rho|^2 |\bar{\Gamma}_0 \bar{\Gamma}_{\rho-f} + \Gamma_0 \bar{\Gamma}_{f-\rho}|^2 d\rho \quad (95)$$

where the notation is again an extension of that used in (79). The power spectrum of the leading nonlinear term is given, as indicated in (80), by the sum of (82), (90), and (95).

The Complete Result

The power spectrum of the output of an AM demodulator when the input is a filtered hybrid AM/PM signal having exponential AM is given by (56), where the leading DC term is given by (68), the leading linear term by (78), and the leading nonlinear term by (80).

REDUCTION TO LINEAR AM

The reduction to linear AM for the basic demodulator output was achieved in Sec. V by identifying the undesired terms $O(a^2)$ in the formula for I_n in (31) and deleting them. This resulted in new expressions (34) for I_n valid for linear AM. In reducing the power spectral results to linear AM it is more convenient to identify and delete the undesired terms in the spectra than to rederive the results using the reduced form for I_n .

Leading DC and Linear Terms

Inasmuch as the reduction to linear AM involves the suppression of terms $O(a^2)$ and these can only appear for I_n , $n > 2$, it is clear that the leading DC and linear terms given by (68) and (70) are the same for both exponential and linear AM.

Leading Nonlinear Term

Of the components of δ_{22} identified in (80), it is seen that only T_1 , T_2 , and T_3 involve I_2 . To reduce these terms to linear AM, the procedure is to note that

$$\begin{aligned} (h_1 + \bar{h}_2)^2 &= [(a_1 + i\varphi_1) + (a_2 - i\varphi_2)]^2 \\ &= a_1^2 + 2a_1a_2 + a_2^2 + 2i(a_1 + a_2)(\varphi_1 - \varphi_2) - (\varphi_1 - \varphi_2)^2 \end{aligned} \quad (96)$$

Thus, the terms a_1^2 and a_2^2 should be suppressed¹ in $(h_1 + \bar{h}_2)^2$. This is done by writing

$$I_2 = \frac{1}{2|\Gamma_0|^2} \iint \gamma_1 \bar{\gamma}_2 [(h_1 + \bar{h}_2)^2 - (a_1^2 + a_2^2)] \quad (97)$$

in the spectral analysis rather than using (31) for I_2 .

Nonlinear Term T_1 . Using (97) for I_2 in (80) leads to

$$\begin{aligned} T_1 &= FE \frac{\delta_0^2}{4|\Gamma_0|^4} \iint \gamma_u \bar{\gamma}_v [(h_{t-u} + \bar{h}_{t-v})^2 - (a_{t-u}^2 + a_{t-v}^2)] du dv \\ &\quad \iint \gamma_u \bar{\gamma}_v [(h_{t+\tau-u} + \bar{h}_{t+\tau-v})^2 - (a_{t+\tau-u}^2 + a_{t+\tau-v}^2)] du dv \\ &= \frac{1}{4|\Gamma_0|^2} F \iiint \gamma_u \bar{\gamma}_v \gamma_w \bar{\gamma}_x E[(h_{t-u} + \bar{h}_{t-v})^2 (h_{t+\tau-w} + \bar{h}_{t+\tau-x})^2 \\ &\quad - (h_{t-u} + \bar{h}_{t-v})^2 (a_{t+\tau-w}^2 + a_{t+\tau-x}^2) - (a_{t-u}^2 + a_{t-v}^2) (h_{t+\tau-w} + \bar{h}_{t+\tau-x})^2 \\ &\quad + (a_{t-u}^2 + a_{t-v}^2) (a_{t+\tau-w}^2 + a_{t+\tau-x}^2)] du dv dw dx \end{aligned} \quad (98)$$

The leading term $E(h_{t-u} + \bar{h}_{t-v})^2 (h_{t+\tau-w} + \bar{h}_{t+\tau-x})^2$ is the complete term

¹The rationale for retaining the $a_1 a_2$ term in (96) is discussed in connection with the reduction of (32) to (33) for linear AM.

for T_1 for exponential AM in (81). Thus, the remaining terms in (98) represent the correction required to reduce the results to linear AM. To determine the form of these correction terms, first note that applying (60) to the appropriate terms in (98), all of which are squared, consistently results in the production of pairs of terms the first of which always represent third-order DC contributions (in the same fashion as in (82) and (87)). For example, the expected value of the first correction term that appears in (98) becomes

$$E(h_{t-u} + \bar{h}_{t-v})^2 a_{t+\tau-w}^2 = E(h_{t-u} + \bar{h}_{t-v})^2 E a_{t+\tau-w}^2 + 2E^2(h_{t-u} + \bar{h}_{t-v}) a_{t+\tau-w} \quad (99)$$

Furthermore, the remaining (i.e., the second) term in the right-hand side of (99) reduces to

$$\begin{aligned} E^2(h_{t-u} + \bar{h}_{t-v}) a_{t+\tau-w} &= E^2[(a_{t-u} + i\phi_{t-u}) + (a_{t-v} - i\phi_{t-v})] a_{t+\tau-w} \\ &= E^2(a_{t-u} + a_{t-v}) a_{t+\tau-w} \end{aligned} \quad (100)$$

because, from (59), $E a \phi = 0$. Thus, h can be replaced by a in the correction terms in (98). Making these adjustments and applying (62), but with the superscript a rather than $+$ or $-$, then brings the correction term in (98) to the form

$$\begin{aligned} \Delta T_1 = - \frac{1}{2|\Gamma_0|^2} F \iiint \gamma_u \bar{\gamma}_v \gamma_w \bar{\gamma}_x [2R_{\tau-w+u}^a R_{\tau-w+v}^a + 2R_{\tau-x+u}^a R_{\tau-x+v}^a + 2R_{\tau-w+u}^a R_{\tau-x+u}^a \\ + 2R_{\tau-w+v}^a R_{\tau-x+v}^a + (R_{\tau-w+u}^a)^2 + (R_{\tau-x+u}^a)^2 + (R_{\tau-w+v}^a)^2 + (R_{\tau-x+v}^a)^2] du dv dw dx \end{aligned} \quad (101)$$

It is apparent that when (65) is used to take the Fourier transforms, the results will display only the spectral products $W_\rho^a W_{f-\rho}^a$. Also, inasmuch as the autocorrelation variables represented by the subscripts in (101) always involve one duplication, using (52) to introduce the transfer function of the equivalent lowpass filter will

cause every correction term to display one factor Γ_0 or $\bar{\Gamma}_0$. Therefore, it is seen that the reduction of (85) to linear AM can be achieved by simply deleting all terms of the form $W_\rho^a W_{f-\rho}^{a*} \Gamma_\alpha \Gamma_\beta \Gamma_\gamma \Gamma_\delta$ that involve a factor $\bar{\Gamma}_0$ or Γ_0 . The result is

$$T_1 = \frac{1}{2|\Gamma_0|^2} \int \left[W_\rho^a W_{f-\rho}^{a*} |\Gamma_\rho \bar{\Gamma}_\rho + \Gamma_{-\rho} \Gamma_{-\rho}|^2 + \left\{ \frac{2W_\rho^a W_{f-\rho}^{a*}}{W_\rho^a W_{f-\rho}^{a*}} \right\} |\Gamma_0 \bar{\Gamma}_f + \Gamma_f \Gamma_0 - \Gamma_\rho \bar{\Gamma}_\rho + \Gamma_{-\rho} \Gamma_{-\rho}|^2 \right] d\rho \quad (102)$$

Nonlinear Terms T_2 and T_3 . Using (31) for I_1 and (97) for I_2 in (80) leads to

$$T_2 = FE \frac{\delta_0^2}{8|\Gamma_0|^6} \iint \gamma_u \bar{\gamma}_v [(h_{t-u} + \bar{h}_{t-v})^2 - (a_{t-u}^2 + a_{t-v}^2)] du dv$$

$$+ \left[\iint \gamma_u \bar{\gamma}_v (h_{t+\tau-u} + \bar{h}_{t+\tau-v}) du dv \right]^2$$

$$- \frac{1}{8|\Gamma_0|^4} \int \cdots \int \gamma_u \bar{\gamma}_v \gamma_w \bar{\gamma}_x \gamma_y \bar{\gamma}_z E[(h_{t-u} + \bar{h}_{t-v})^2 (h_{t+\tau-w} + \bar{h}_{t+\tau-x})(h_{t+\tau-y} + \bar{h}_{t+\tau-z})$$

$$- (a_{t-u}^2 + a_{t-v}^2)(h_{t+\tau-w} + \bar{h}_{t+\tau-x})(h_{t+\tau-y} + \bar{h}_{t+\tau-z})] du \cdots dz \quad (103)$$

The leading term $E(h_{t-u} + \bar{h}_{t-v})^2 (h_{t+\tau-w} + \bar{h}_{t+\tau-x})(h_{t+\tau-y} + \bar{h}_{t+\tau-z})$ is the complete term for T_2 for exponential AM in (86). Thus, the remaining terms in (103) represent the correction required to reduce the results to linear AM. Using an argument similar to that following (98) and recalling that T_2 and T_3 had mirror symmetry in frequency, it follows that the correction term can be brought into the form

$$\begin{aligned} \Delta(T_2 + T_3) = & - \frac{1}{4|\Gamma_0|^4} F \int \cdots \int \gamma_u \bar{\gamma}_v \gamma_w \bar{\gamma}_x \gamma_y \bar{\gamma}_z (R_{\tau-w+u}^a R_{\tau-y+u}^a \\ & + R_{\tau-w+u}^a R_{\tau-z+u}^a + R_{\tau-x+u}^a R_{\tau-y+u}^a + R_{\tau-x+u}^a R_{\tau-z+u}^a + R_{\tau-w+v}^a R_{\tau-y+v}^a \\ & + R_{\tau-w+v}^a R_{\tau-z+v}^a + R_{\tau-x+v}^a R_{\tau-y+v}^a + R_{\tau-x+v}^a R_{\tau-z+v}^a) du \cdots dz \quad (104) \end{aligned}$$

As was the case with ΔT_1 in (101), applying (65) to take the Fourier transform will result in the production of only the spectral products $W_{\rho}^a W_{f-\rho}^a$. Also, inasmuch as (104) is a sixfold integral and the autocorrelation variables represented by the subscripts in (104) always involve only three variables, it follows that when (52) is used to introduce the transfer function of the equivalent lowpass filter, every correction term will display three factors Γ_0 or $\bar{\Gamma}_0$. Therefore, it is seen that the reduction of (90) to linear AM can be achieved by simply deleting all terms of the form $W_{\rho}^a W_{f-\rho}^a \Gamma_{\alpha} \Gamma_{\beta} \Gamma_{\gamma} \Gamma_{\delta} \Gamma_{\epsilon} \Gamma_{\zeta}$ that involve exactly three factors Γ_0 or $\bar{\Gamma}_0$. The result is

$$\begin{aligned} T_2 + T_3 = & \frac{1}{2|\Gamma_0|^2} \int \left[W_{\rho}^a W_{f-\rho}^a \text{Re}(\Gamma_{-\rho} \bar{\Gamma}_f + \Gamma_{\rho} \bar{\Gamma}_f)(\Gamma_0 \bar{\Gamma}_{-\rho} + \Gamma_{\rho} \bar{\Gamma}_0)(\Gamma_0 \bar{\Gamma}_{\rho-f} + \Gamma_{f-\rho} \bar{\Gamma}_0) \right. \\ & \left. + \left\{ \frac{2W_{\rho}^a}{W_{\rho}^c} \frac{W_{f-\rho}^c}{W_{f-\rho}^f} \right\} \text{Re}(\Gamma_{-f} \bar{\Gamma}_0 + \Gamma_0 \bar{\Gamma}_f - \Gamma_{\rho} \bar{\Gamma}_{f-\rho} - \Gamma_{\rho} \bar{\Gamma}_f)(\Gamma_0 \bar{\Gamma}_{-\rho} + \Gamma_{\rho} \bar{\Gamma}_0)(\Gamma_0 \bar{\Gamma}_{\rho-f} - \Gamma_{f-\rho} \bar{\Gamma}_0) \right] d\rho \quad (105) \end{aligned}$$

The power spectrum of the leading nonlinear term is given by the linear-AM version of (80); that is

$$\delta_2 \otimes \delta_2 = T_1 + (T_2 + T_3) + T_4 \quad (106)$$

See (102), (105), and (95) for the appropriate formulations for T_1 , $T_2 + T_3$, and T_4 .

The Complete Result

The power spectrum $W(f)$ of the output of an AM demodulator when the input is a filtered hybrid AM/PM signal having linear AM is given by (56), where the leading DC term is given by (68), the leading linear term by (78), and the leading nonlinear term by (106). These are repeated here for convenience

$$W(f) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i!j!} \delta_i \otimes \delta_j \quad (56)$$

where the significant terms are given by

$$\underline{i + j = 0}: \quad \delta_0 \otimes \delta_0 = FE |\Gamma_0|^2 = |\Gamma_0|^2 \delta(0) \quad \underline{\text{Leading DC term}} \quad (68)$$

$$\delta_1 \otimes \delta_1 = \frac{1}{4 |\Gamma_0|^2} \left\{ \frac{W_f^a}{W_f^p} \right\} |\bar{\Gamma}_0 \Gamma_f \pm \Gamma_0 \bar{\Gamma}_{-f}|^2 \cdot \underline{\text{Leading linear term}} \quad (78)$$

$$\delta_2 \otimes \delta_2 = T_1 - (T_2 + T_3) + T_4 \quad \underline{\text{Leading nonlinear term}} \quad (106)$$

and

$$T_1 = \frac{1}{2|\Gamma_0|^2} \int \left[W_\rho^a W_{f-\rho}^a |\Gamma_{\rho-f} \Gamma_\rho + \Gamma_{-\rho} \Gamma_{f-\rho}|^2 \right. \\ \left. + \left\{ \frac{2W_\rho^a}{W_\rho^\phi} \frac{W_{f-\rho}^\phi}{W_{f-\rho}^\phi} \right\} |\Gamma_0 \Gamma_{f-\rho} + \Gamma_{-\rho} \Gamma_0 - \Gamma_{\rho-f} \Gamma_\rho + \Gamma_{-\rho} \Gamma_{f-\rho}|^2 \right] d\rho \quad (102)$$

$$T_2 + T_3 = \frac{1}{2|\Gamma_0|^2} \int \left[W_\rho^a W_{f-\rho}^a \text{Re}(\Gamma_{-\rho} \Gamma_{f-\rho} + \Gamma_{\rho-f} \bar{\Gamma}_\rho)(\Gamma_0 \bar{\Gamma}_{-\rho} + \Gamma_\rho \bar{\Gamma}_0)(\Gamma_0 \bar{\Gamma}_{\rho-f} + \Gamma_{f-\rho} \bar{\Gamma}_0) \right. \\ \left. + \left\{ \frac{2W_\rho^a}{W_\rho^\phi} \frac{W_{f-\rho}^\phi}{W_{f-\rho}^\phi} \right\} \text{Re}(\Gamma_{-f} \bar{\Gamma}_0 + \Gamma_0 \bar{\Gamma}_{-f} - \Gamma_{\rho-f} \bar{\Gamma}_{-\rho} + \Gamma_\rho \bar{\Gamma}_{f-\rho})(\Gamma_0 \bar{\Gamma}_{-\rho} + \Gamma_\rho \bar{\Gamma}_0)(\Gamma_0 \bar{\Gamma}_{\rho-f} + \Gamma_{f-\rho} \bar{\Gamma}_0) \right] d\rho \quad (105)$$

$$T_4 = \frac{1}{8|\Gamma_0|^6} \int \left\{ \frac{W_\rho^a}{2W_\rho^\phi} \frac{W_{f-\rho}^a}{W_{f-\rho}^\phi} \right\} |\bar{\Gamma}_0 \Gamma_{-\rho} + \Gamma_0 \bar{\Gamma}_\rho|^2 |\bar{\Gamma}_0 \Gamma_{\rho-f} + \Gamma_0 \bar{\Gamma}_{f-\rho}|^2 d\rho \quad (95)$$

$\delta(0)$ in (68) is the Dirac delta function. The subscripts are the arguments of the functions to which they are appended. W^a and W^ϕ are the power spectra of the amplitude and phase modulation functions, respectively. The matrix-multiplication-like notation in (105) and (95) denotes, from (79),

$$\begin{Bmatrix} A \\ B \end{Bmatrix} (a \pm b) = A(a \pm b) + B(a - b) \quad (79a)$$

and

$$\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} (a \pm b)(c \pm d) = A(a+b)(c+d) + B(a+b)(c-d) + C(a-b)(c-d)$$

SPECIALIZATION OF LINEAR AM TO A PROPERLY TUNED, NARROW, SYMMETRIC, BANDPASS FILTER

The spectral results for linear AM will be specialized to the case when the narrowband filter is nearly symmetric and properly tuned. This case is of interest because it corresponds to the situation where an ideal AM receiver using an envelope detector is properly tuned to a hybrid AM/PM signal. Also it is a case in which nonlinear terms simplify enough to permit some interpretation. The specialization is done by noting that the impulse response of the equivalent lowpass filter is very nearly real for this case; that is $\theta(t) \cong 0$ in (11). Thus, its transfer function Γ_f displays the conjugate symmetry property (9) of the filter itself so $\Gamma_f = \bar{\Gamma}_{-f}$ and Γ_0 is real.

The leading DC term (68) is, of course, not affected. The leading linear term (78) demonstrates an interesting symmetry even before specialization to a properly tuned, symmetric filter. It is seen from (79) that W_f^a , the spectrum of the amplitude modulating signal, is multiplied by the square of the magnitude of the sum of $\bar{\Gamma}_0 \Gamma_f$ and $\Gamma_0 \bar{\Gamma}_{-f}$, whereas W_f^ϕ , the spectrum of the phase modulating signal, is multiplied by the square of the magnitude of their difference. Thus, the appearance of the phase modulated signal in the output of an AM demodulation is related to the lack of symmetry in the filter. In the limit, as $\bar{\Gamma}_0 \Gamma_f \Rightarrow \Gamma_0 \bar{\Gamma}_{-f}$, the phase modulation disappears and (78) becomes

$$\delta_1 \otimes \delta_1 = W_f^a |\Gamma_f|^2 \quad (107)$$

which agrees with the expansion given by (35) where it was noted that the linear term δ_1 consists of the amplitude modulation filtered by the equivalent lowpass filter. The result (107) is valid for both exponential and linear AM.

For the case of exponential AM, the leading nonlinear term (80) simplifies somewhat but the results are of little practical interest and are not shown here. However, for the case of linear AM the simplification of the leading nonlinear term (106) is dramatic. Letting $\Gamma_f = \bar{\Gamma}_{-f}$ and Γ_0 be real in (102), (105), and (95) leads to

$$\delta_2 \otimes \delta_2 = \frac{2}{\Gamma_0^2} \int W_\rho^\varphi W_{f-\rho}^\varphi | \Gamma_0 \bar{\Gamma}_f - \Gamma_\rho \bar{\Gamma}_{\rho-f} |^2 d\rho \quad (108)$$

which is in agreement with the expansion given by (35) where it was noted that the leading nonlinear term δ_2 represents distortion produced by the phase modulation alone. A simple interpretation of (108) does not appear possible but the form of a difference between $\Gamma_0 \bar{\Gamma}_f$ and $\Gamma_\rho \bar{\Gamma}_{\rho-f}$ suggests that the integration on ρ is a conditioning of the phase modulation spectrum over the filter dissymmetry as a function of all possible center frequencies. That is, the wider the filter is with respect to the bandwidth of the phase modulation, and the flatter its response, the less the value of (108) will be.

VIII. CONCLUSIONS

The analysis presented in this Note is purely theoretical. The intention was to develop the formulations that can be used in subsequent numerical calculations. Principal interest was focused on the output of an AM receiver, which may be mistuned and whose spectral response characteristics may not be symmetrical, when the input is a conventional AM signal that is simultaneously phase modulated. An expansion of the output time function is given in Sec. V and the output power spectrum in Sec. VII. An expansion of the output time function from a PM receiver is given in Sec. VI for completeness but the corresponding output power spectrum remains to be derived.

The situation of practical interest is the one in which the undistorted output (i.e., the linear component), which is given by the leading term, is large in comparison with the distortion, which is conveyed by the remaining terms in the output time-function and power-spectrum expansions. Thus, the principal contribution to the distortion is contained in the leading distortion term if the input modulating signals are well behaved and the expansions converge rapidly. The output time-function expansion probably converges for modulating signals of practical interest. The situation is less clear with respect to output power-spectrum expansion because it was necessary to assume that the modulating signals were independent random gaussian processes to make the analysis tractable. Some questions concerning convergence are discussed in Ref. 3 but the matter is by no means resolved. Thus, the expansion must be used with care; it is undoubtedly valid only for the case of small distortion and should be verified experimentally, if possible.

Appendix

THE ANALYTIC FILTER

Consider a real physically realizable filter having an impulse response $g(t)$ and transfer function $G(f)$ defined by (7) and (8), respectively. An equivalent analytic filter can be defined having an impulse response

$$u(t) = \frac{1}{2} [g(t) + i\hat{g}(t)] \quad (\text{A.1})$$

where $\hat{g}(t)$ is the Hilbert transform of $g(t)$. [5] If the Fourier transform F of $g(t)$ is given

$$Fg(t) = G(f) \quad (\text{A.2})$$

then

$$F\hat{g}(t) = -i \operatorname{sgn} f G(f) \quad (\text{A.3})$$

where

$$\operatorname{sgn} f = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases} \quad (\text{A.4})$$

Thus, the analytic impulse response $u(t)$ is that of a filter having a transfer function

$$U(f) = \begin{cases} G(f), & f > 0 \\ 0, & f \leq 0 \end{cases} \quad (\text{A.5})$$

Inasmuch as the analytic filter has the same transfer function as its equivalent real filter for positive frequencies, its response to analytic signals, which contain only positive frequencies, is identical to that of the real filter. That is, if $s(t)$ is the input signal, then

$$\int_0^\infty g(\tau)[s(t - \tau) + i\hat{s}(t + \tau)]d\tau = \frac{1}{2} \int_{-\infty}^\infty [g(\tau) + i\hat{g}(\tau)][s(t - \tau) + i\hat{s}(t - \tau)]d\tau \quad (A.6)$$

which can be verified by taking the Fourier transform of both sides.

The analytic filter is not the same as the filter having the complex impulse response $\gamma(t)$ given by (11) because the real and imaginary parts of $\gamma(t)$ are not a Hilbert pair.[8] That is,

$$r(t)\sin[2\pi f_c t + \theta(t)] \neq Hr(t)\cos[2\pi f_c t + \theta(t)] \quad (A.7)$$

where H denotes the Hilbert transform. However, the relationship is approximately true for a narrow bandpass filter, which is the reason why the response $\bar{\Gamma}(f)$ of the complex narrow bandpass filter can be assumed to be negligible for negative frequencies.[9]

It should be noted that the analytic filter is realizable in the sense that the real part of its impulse response is the causal impulse response of a physically realizable filter. The imaginary part of its impulse response, which is the Hilbert transform of the real causal impulse response, is not causal in general; hence, the extended lower limit on the right-hand integral in (A.6). Using the analytic filter may be a nuisance for time-domain analysis because of the need to find the Hilbert transform. Its use in the frequency domain, on the other hand, is simple because it merely entails deleting the transfer function of the real filter for negative frequencies.

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